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CELESTIAL MECHANICS AND ASTRODYNAMICS

A TRAJECTORY OPTIMIZATION TECHNIQUE BASED UPON THE THEORY OF THE SECOND VARIATION

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Abstract

A successive approximation method based upon the theory of the second variation is developed. In the early stage of computation, the process behaves much like the gradient/penalty function process with boundary conditions met only approximately. In the terminal stage, convergence more rapid than that of a gradient method is achieved with "exact" satisfaction of boundary conditions an integral part of the process. Since the equations of variation of the Euler-Lagrange equations are employed in the computational scheme, only slight additional effort is required to perform a check of the generalized Jacobi (Mayer) condition.

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Introduction

Research in methods for numerically determining optimal trajectories has taken, in the main, two directions: study of steep descent processes in various versions,[1-8] and development of iterative solution schemes for the Euler-Lagrange equations.[9-19]

The strong points of steep descent processes are that convergence does not depend upon availability of a good initial estimate of the optimal trajectory as a starting point, and that they seek out weak relative minima as distinct from points at which the functional is merely stationary. The main weakness in practical applications is that convergence slows in the terminal phase of the process as the optimal trajectory is approached. As

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with other methods, there is difficulty if either the Legendre-Clebsch condition or the generalized Jacobi condition is met only marginally, i.e., if the solution exhibits either singular subarcs or conjugate endpoints, and this appears in the form of convergence so poor that, practically speaking, the method fails. Except in these cases, an a posteriori check of the Weierstrass condition will establish the weak relative minimum obtained by the process as a strong relative minimum if the strengthened form of the condition is met.

Iterative solution of the Euler equations requires a good first estimate of multiplier initial conditions in order to converge at all. Convergence of the process is assured, theoretically, if the trial initial conditions are sufficiently close and if the solution is nonsingular with nonconjugate endpoints. The establishment of minimality requires separate checks of the Weierstrass and generalized Jacobi conditions. The latter is somewhat complex computationally and has been only rarely performed in practice. Convergence as a practical matter is troublesome, particularly so in the case of atmospheric flight of lifting vehicles.[9] Some success has been realized in the initial value iteration approach in the computation of optimal rocket trajectories in vacuum.[11-13] Even in this class of comparatively well-behaved Euler solutions, the main practical difficulty is in obtaining a trial solution whose end conditions approximate those desired. An attractive feature of the method is fast convergence in the terminal phase of the computation. A refinement of the method is the use of a linearized version of the Euler equations to obtain the elements of the transition matrix needed in the iteration.[14,15] In the work of Ref. 15, a separate computation via gradient method was employed to obtain the first estimate of the multiplier initial values. A different sort of iteration scheme tailored to "bang-bang" control problems is reported in Ref. 16. A method based upon the Euler-Lagrange equations and a generalization of Newton's method has been investigated in Refs. 17, 18, and 19, but has received little attention in trajectory applications. The relationship of this method to that which we discuss herein will be examined in the latter portion of the paper.

In the present paper, we present a successive approximation technique based upon the theory of the second variation. As with gradient methods, the initial trajectory estimate is required to be neither optimal nor necessarily a good approximation. In the initial phase of computation, the penalty function treatment of terminal conditions is employed and the behavior of the process strongly resembles that of a gradient/penalty function process as a result of step size constraints being operative which limit the amount of improvement sought during each cycle. These constraints are progressively relaxed, finally dropped, and the terminal penalty scheme discarded in favor of "exact" terminal conditions that are ultimately satisfied if a solution exists, in the sense that the specified conditions are attainable. This second phase of the process is computationally similar to iteration on the

Euler equations and shares the feature of fast terminal convergence.

Problem Formulation and Penalty Function Approximation

We begin with a statement of the trajectory optimization problem in the usual Mayer format. Given a system of first-order differential equations

$$\dot{x}_i = g_i(x_1, \ldots, x_n, y_1, \ldots, y_\ell, t) \tag{1}$$

it is required to find a solution of this system satisfying certain specified initial and terminal conditions and providing a minimum of some function $P = P(x_{1_f}, \ldots, x_{n_f}, t_f)$ of the terminal values of the variables x_i and the terminal time. The variables x_i, $i = 1, \ldots, n$, are state variables and the y_k, $k = 1, \ldots, \ell$, control variables. The latter may be subject to inequality constraints of the form

$$y_{k1} \leq y_k \leq y_{k2} \qquad k = 1, \ldots, \ell \tag{2}$$

as subsidiary conditions of the problem. We will deal primarily with the relatively simple case in which such inequality constraints are absent, adding some comments in the latter portion of the paper on the treatment of inequalities.

For simplicity of presentation, we will assume that all of the initial values of the x_i are fixed at a specified initial time t_o:

$$x_i(t_o) = \tilde{x}_{i_o} \qquad i = 1, \ldots, n \tag{3}$$

The terminal time t_f will be regarded as unspecified, which is more often the case than not in applications. The terminal values of the first m of the x_i will be taken as fixed:

$$x_i(t_f) = \tilde{x}_{i_f} \qquad i = 1, \ldots, m \tag{4}$$

and those of the remaining ones unspecified. Some or all of the terminal values of the x_i, $i = m + 1, \ldots, n$ may appear as arguments of the function P whose minimum is sought. The particular form of boundary conditions chosen here for definiteness is reasonably typical of problems arising in applications, and, in any case, modification of the ensuing analysis to accommodate other types of boundary conditions will present no essential difficulty.

An alternate formulation of the problem is given in terms of an approximation employing an augmented function of the terminal values:

$$P'(x_{1_f},\ldots,x_{n_f},t_f) = P(x_{m+1_f},\ldots,x_{n_f},t_f) + \frac{1}{2}\sum_{j=1}^{m} K_j(x_{j_f} - \tilde{x}_{j_f})^2 \tag{5}$$

A minimum of the function P' is to be sought without specification on the terminal values of the x_i. With $K_j > 0$, $j = 1, \ldots, m$, the second member of (5), which may be termed a "penalty function," will be positive if there are deviations from the desired terminal values $\tilde{x}_{j_f}$. If the K_j are chosen to be numerically large, it may be anticipat i that a trajectory, optimal in the sense of minimizing P', will come close to meeting the desired terminal conditions, provided, of course, that these are attainable. One advantage of a penalty function treatment of terminal conditions is that a solution of the problem may be computed even though the desired terminal conditions are unattainable, i.e., even if no solution exists for the corresponding problem stated in terms of fixed terminal conditions. In such cases, the resulting solution, which ails to closely approximate the desired terminal conditions, may be of considerable value to the analyst in establishing physically reasonable terminal specifications for families of solutions, information which is available a priori only rarely. The basis, genesis, and application of the penalty function technique are discussed in Ref. 1.

An Expansion about a Reference Trajectory

In the classical theory of the Mayer problem, the constraints given by the differential equations (1) are adjoined to the functional P' by means of Lagrange multipliers:

$$J = P'(x_{1_f},\ldots,x_{n_f},t_f) + \int_{t_o}^{t_f} \sum_{i=1}^{n} \lambda_i(-\dot{x}_i + g_i)dt \tag{6}$$

and an expansion of J is performed in the neighborhood of a reference trajectory $\bar{x}_i(t)$, $\bar{y}_k(t)$:

$$J = J_o + J_1 + \tfrac{1}{2} J_2 + \ldots \tag{7}$$

Here, $J_o = P'(\bar{x}_{1_f}, \ldots, \bar{x}_{n_f}, \bar{t}_f)$, since the integrand in (6) vanishes along the reference trajectory, which is presumed to satisfy the system (1). J_1 and J_2 are, respectively, the collections of first- and second-order terms in the variations δx_i, δy_k of the state and control variables from those of the reference trajectory; they are known as the first and second variations. Since the analytical form of the second variation J_2 appearing in the classical literature corresponds to the case of a reference trajectory that satisfies the Euler-Lagrange equations of the problem as well as the system (1), the following derivation of the slightly more general form corresponding to a reference trajectory that satisfies only (1) is needed.

An expansion of the function P' in the neighborhood of the terminal point of the reference trajectory is given by

$$\begin{aligned} P' = P'(\bar{x}_{1_f}, \ldots, \bar{x}_{n_f}, \bar{t}_f) &+ \\ &\sum_{i=1}^{n} \frac{\partial P'}{\partial x_{i_f}} \Delta x_{i_f} + \frac{\partial P'}{\partial t_f} \delta t_f + \\ &\tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 P'}{\partial x_{i_f} \partial x_{j_f}} \Delta x_{i_f} \Delta x_{j_f} + \\ &\sum_{i=1}^{n} \frac{\partial^2 P'}{\partial x_{i_f} \partial t_f} \Delta x_{i_f} \delta t_f + \\ &\tfrac{1}{2} \frac{\partial^2 P'}{\partial t_f^2} \delta t_f^2 + \ldots \end{aligned} \tag{8}$$

Since we desire ultimately to obtain an approximation from (8) valid to second order in control variations $\delta y_k(t)$, evidently we must employ estimates of the state variable terminal increments Δx_{i_f} which are correct to second order in the control variations, at least in the first-order terms of (8). A first-order estimate of the variations in the state variables is given by the solution of a linearized version of (1):

$$\delta\dot{x}_i = \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j}\,\delta x_j + \sum_{k=1}^{\ell} \frac{\partial g_i}{\partial y_k}\,\delta y_k \tag{9}$$

$$i = 1, \ldots, n$$

$$\delta x_i(t_o) = 0 \tag{10}$$

in which the partial derivatives of the functions g_i are evaluated along the reference trajectory. An alternate and equivalent first-order estimate of δx_i at the terminal time of the reference trajectory $\bar{t}_f$ is given by

$$\delta x_i(\bar{t}_f) = \int_{t_o}^{\bar{t}_f} \sum_{k=1}^{\ell} \frac{\partial H_i}{\partial y_k}\,\delta y_k \, dt \tag{11}$$

The function $H_i \equiv \sum_{j=1}^{n} \lambda_j^{(i)} g_j$ is defined in terms of that solution $\lambda_j^{(i)}$ of the adjoint system

$$\dot{\lambda}_j = - \sum_{i=1}^{n} \lambda_i \frac{\partial g_i}{\partial x_j} \qquad j = 1, \ldots, n \tag{12}$$

which corresponds to the special boundary conditions

$$\begin{aligned} \lambda_j(\bar{t}_f) &= 1 \qquad & j = i \\ &= 0 \qquad & j \neq i \end{aligned} \tag{13}$$

A second-order estimate of the increment in x_i at $t = \bar{t}_f$ is given by the integral

$$\xi_i(\bar{t}_f) = \int_{t_o}^{\bar{t}_f} \left(\sum_{k=1}^{\ell} \frac{\partial H_i}{\partial y_k}\,\delta y_k + \omega_i \right) dt \tag{14}$$

where

$$2\omega_i \equiv \sum_{p=1}^{n}\sum_{q=1}^{n} \frac{\partial^2 H_i}{\partial x_p\, \partial x_q}\,\delta x_p\, \delta x_q + 2\sum_{p=1}^{n}\sum_{k=1}^{\ell}\frac{\partial^2 H_i}{\partial x_p\, \partial y_k}\,\delta x_p\, \delta y_k + \sum_{k=1}^{\ell}\sum_{s=1}^{\ell}\frac{\partial^2 H_i}{\partial y_k\, \partial y_s}\,\delta y_k\, \delta y_s \tag{15}$$

The second member of the integrand of (14) utilizes the influence functions of (12) and (13) to obtain an estimate of the effects of the second-order terms in the g_i that were omitted in (9). A corresponding estimate for the increment in x_i at a variable terminal time $t_f = \bar{t}_f + \delta t_f$ is given by

$$\Delta x_{i_f} = \xi_{i_f} + \left(\bar{g}_{i_f} + \sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}}\,\delta x_{j_f} + \sum_{k=1}^{\ell}\frac{\partial g_{i_f}}{\partial y_{k_f}}\,\delta y_{k_f}\right)\delta t_f + \frac{1}{2}\left(\sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}}\,\bar{g}_{j_f} + \sum_{k=1}^{\ell} \frac{\partial g_{i_f}}{\partial y_{k_f}}\,\dot{\bar{y}}_{k_f} + \frac{\partial g_{i_f}}{\partial t_f}\right)\delta t_f^{2} \tag{16}$$

in which the abbreviations $\delta x_{i_f} \equiv \delta x_i(\bar{t}_f)$, $\xi_{i_f} = \xi_i(\bar{t}_f)$, $\bar{g}_{i_f} \equiv g_i(\bar{x}_{1_f}, \ldots, \bar{x}_{n_f}, \bar{y}_{1_f}, \ldots, \bar{y}_{\ell_f}, \bar{t}_f)$ are employed.

Substituting (16) into (8), and discarding terms of order higher than second, we then obtain the desired second-order approximation to P':

$$
\begin{aligned}
J_0 + J_1 + \tfrac{1}{2} J_2 = {} & P'(\bar{x}_{1_f}, \ldots, \bar{x}_{n_f}, \bar{t}_f) + \\
& \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{i_f}} \Bigg[\xi_{i_f} + \Bigg(g_{i_f} + \sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta x_{j_f} + \\
& \sum_{k=1}^{\ell} \frac{\partial g_{i_f}}{\partial y_{k_f}} \delta y_{k_f} \Bigg) \delta t_f \quad \tfrac{1}{2} \Bigg(\sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \bar{g}_{j_f} + \\
& \sum_{k=1}^{\ell} \frac{\partial g_{i_f}}{\partial y_{k_f}} \dot{\bar{y}}_{k_f} + \frac{\partial g_{i_f}}{\partial t_f} \Bigg) \delta t_f^{\,2} \Bigg] + \frac{\partial P'}{\partial t_f} \delta t_f + \\
& \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 P'}{\partial x_{i_f} \partial x_{j_f}} (\delta x_{i_f} + \bar{g}_{i_f} \delta t_f) \cdot \\
& \qquad (\delta x_{j_f} + \bar{g}_{j_f} \delta t_f) + \\
& \sum_{i=1}^{n} \frac{\partial^2 P'}{\partial x_{i_f} \partial t_f} (\delta x_{i_f} + \bar{g}_{i_f} \delta t_f) \delta t_f + \\
& \tfrac{1}{2} \frac{\partial^2 P'}{\partial t_f^{\,2}} \delta t_f^{\,2}
\end{aligned}
\tag{17}
$$

Here, δx_{i_f} and δt_f have been regarded as first-order quantities and the difference between ξ_{i_f} and δx_{i_f} as of second order.

It is of interest to relate this expression to the classical development in which the reference trajectory is an Euler solution, for the case of open terminal conditions on the state variables and the terminal time. The functions H and ω appearing in the classical development are

$$
H = \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{i_f}} H_i \tag{18}
$$

$$\omega = \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{i_f}} \omega_i \tag{19}$$

and the multiplier functions are corresponding linear combinations of the fundamental solutions $\lambda_j^{(i)}$ of the system (12) defined by the unit terminal conditions (13). If inequality constraints on the control variables y_k are absent, the Euler-Lagrange equations for the control variables are

$$\frac{\partial H}{\partial y_k} = 0 \qquad k = 1, \ldots, \ell \tag{20}$$

and certain terms in (17) which contain $\delta y_k(t)$, δy_{k_f}, and $\dot{y}_{k_f}$ disappear.

Successive Approximation Process

We consider the possibility of determining control increments $\delta y_k(t)$ that minimize the second-order approximation to P' given by (17). This variational problem is of Bolza form owing to the appearance of the integral

$$\sum_{i=1}^{n} \frac{\partial P'}{\partial x_{i_f}} \xi_{i_f} = \int_{t_o}^{\bar{t}_f} \left(\frac{\partial H}{\partial y_k} \delta y_k + \omega\right) dt \tag{21}$$

in the expression (17). As subsidiary conditions of the variational problem, we have the system of differential equations (9) which defines the $\delta x_i(t)$. If the reference trajectory satisfies the specified initial conditions, which we shall assume to be the case, the appropriate initial conditions on the δx_i are $\delta x_i(t_o) = 0$. Terminal conditions on the δx_i are unspecified, as is the increment in terminal time δt_f. The quadratic/linear format of this Bolza variational problem is computationally attractive, and this provides a primary motivation for the approach to the successive approximation process presently under consideration.

Adjoining the differential constraints (9) with Lagrange multipliers $\delta\lambda_i$, $i = 1, \ldots, n$, we proceed to write the Euler-Lagrange equations, the Weierstrass necessary condition, and the transversality conditions for the problem. The Euler-Lagrange equations corresponding to the state variables are:

$$\delta\dot{\lambda}_i = - \sum_{j=1}^{n} \delta\lambda_j \frac{\partial g_j}{\partial x_i} - \frac{\partial}{\partial \delta x_i} \omega \qquad (22)$$

$$i = 1, \ldots, n$$

The Weierstrass necessary condition takes the form

$$h(\delta y_1^*, \ldots, \delta y_\ell^*) \geq h(\delta y_1, \ldots, \delta y_\ell) \qquad (23)$$

in which the δy_k^*, $k = 1, \ldots, \ell$, are arbitrary. The function h is given by

$$h = \sum_{i=1}^{n} \delta\lambda_i \left(\sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} \delta x_j + \sum_{k=1}^{\ell} \frac{\partial g_i}{\partial y_k} \delta y_k \right) + \sum_{k=1}^{\ell} \frac{\partial H}{\partial y_k} \delta y_k + \omega \qquad (24)$$

Owing to the linear/quadratic form taken by h, the Weierstrass necessary condition is equivalent to the Legendre-Clebsch necessary condition and the Euler equations for the δy_k. This would not be the case if considerations included inequality constraints on the control variables.

The transversality conditions corresponding to open δx_{j_f} are

$$\sum_{i=1}^{n} \frac{\partial P'}{\partial x_{i_f}} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta t_f + \sum_{i=1}^{n} \frac{\partial^2 P'}{\partial x_{i_f} \partial x_{j_f}} (\delta x_{i_f} + \bar{g}_{i_f} \delta t_f) + \frac{\partial^2 P'}{\partial x_{j_f} \partial t_f} \delta t_f - \delta\lambda_{j_f} = 0 \qquad j = 1, \ldots, n \qquad (25)$$

and to open δt_f

$$\sum_{i=1}^{n} \frac{\partial P'}{\partial x_{i_f}} \bar{g}_{i_f} + \frac{\partial P'}{\partial t_f} + \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{i_f}} \left[\left(\sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta x_{j_f} + \sum_{k=1}^{\ell} \frac{\partial g_{i_f}}{\partial y_{k_f}} \delta y_{k_f} \right) + \right.$$

$$\left(\sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \bar{g}_{j_f} + \sum_{k=1}^{\ell} \frac{\partial g_{i_f}}{\partial y_{k_f}} \dot{\bar{y}}_{k_f} + \frac{\partial g_{i_f}}{\partial t_f}\right) \delta t_f \Big] +$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 P'}{\partial x_{i_f} \partial x_{j_f}} \bar{g}_{j_f} (\delta x_{i_f} + \bar{g}_{i_f} \delta t_f) + \tag{26}$$

$$\sum_{i=1}^{n} \frac{\partial^2 P'}{\partial x_{i_f} \partial t_f} (\delta x_{i_f} + 2\bar{g}_{i_f} \delta t_f) + \frac{\partial^2 P'}{\partial t_f^2} \delta t_f = 0$$

The analysis of our successive approximation process bears a resemblance to that of the classical accessory minimum problem for the second variation. In the classical analysis, the reference trajectory satisfies the Euler-Lagrange equations and the transversality conditions for the problem of minimizing P'; the requirement of positive semidefiniteness for the second variation suggests the problem of minimizing the second variation, the so-called accessory minimum problem. The analysis leads to the Legendre-Clebsch and generalized Jacobi (Mayer) necessary conditions. A main feature of the analysis, in the absence of inequality constraints on the control variables, is that the Euler-Lagrange equations and transversality conditions of the accessory problem are precisely linearized versions of those for the original problem of minimizing P'. Such is also the case in the present analysis, with the slight but important difference that certain zero-order terms remain in the linearized expressions due to the nonoptimality of the reference trajectory. A somewhat analogous approach, employing an optimal reference trajectory, has been taken in Ref. 20 in connection with an optimal guidance approximation scheme.

In the present application - the determination of an optimal trajectory through successive improvements on a nonoptimal reference trajectory - there is a question concerning the existence of a minimum of (17) and a related question concerning the convergence of the process. If the reference trajectory were close to the optimal trajectory sought, existence and convergence arguments of sorts could be built around this fact. Such a requirement on the reference trajectory chosen as a starting point for the computational process would, however, obviously represent an undesirable restriction. On the other hand, if the reference trajectory satisfies only the basic system (1) and the initial conditions, but is otherwise arbitrary, there is no assurance that a minimum of (17) exists, and, in fact, it will commonly be the case that (17) is unbounded below. If, for example, the function h given by (24) has no minimum in the δy_k, the question of existence of a minimum of the approximation is settled in the negative. Thus, at least in the early phase of the computational process, it appears necessary to introduce restrictions on the "step size" as measured in terms of the norms of the functions $\delta y_k(t)$. We therefore alter the problem by the introduction of additional subsidiary conditions given by the equations

$$\delta \dot{x}_{n+k} = \tfrac{1}{2}\,\delta y_k^{\,2} \qquad \delta x_{n+k}(t_o) = 0 \qquad k = 1, \ldots, \ell \tag{27}$$

defining variables δx_{n+k} whose terminal values $\delta x_{n+k}(t_f)$ are integral square measures of the magnitudes of the control variable increments $\delta y_k(t)$.

With the constraints (27) adjoined by means of additional multipliers $\delta\lambda_i$, $i = n+1, \ldots, n+\ell$, the analysis proceeds as before, and the Euler-Lagrange equations (22) and the transversality conditions (25) and (26) are unchanged. The Euler-Lagrange equations corresponding to the variables δx_{n+k}, $k = 1, \ldots, \ell$ are

$$\delta \dot{\lambda}_{n+k} = 0 \qquad\qquad k = 1, \ldots, \ell \tag{28}$$

indicating the constancy of these multipliers. The Weierstrass necessary condition is given by

$$\hat{h}(\delta y_1^{\,*}, \ldots, \delta y_\ell^{\,*}) \geq \hat{h}(\delta y_1, \ldots, \delta y_\ell) \tag{29}$$

in which the $\delta y_k^{\,*}$ are arbitrary, and the function $\hat{h}$ is given by

$$\hat{h} = \sum_{i=1}^{n} \delta\lambda_i \left(\sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j}\,\delta x_j + \sum_{k=1}^{\ell} \frac{\partial g_i}{\partial y_k}\,\delta y_k \right) + \sum_{k=1}^{\ell} \frac{\partial H}{\partial y_k}\,\delta y_k + \omega + \tfrac{1}{2} \sum_{k=1}^{\ell} \delta\lambda_{n+k}\,\delta y_k^{\,2} \tag{30}$$

the last member arising from the additional constraints (27).

The Weierstrass necessary condition (29) provides information of value in the choice of the constraint multipliers $\delta\lambda_i$, $i = n+1, \ldots, n+\ell$, this choice being equivalent to the establishment of the step size parameters $\delta x_i(t_f)$, $i = n+1, \ldots, n+\ell$. In the case of an unbounded control variable y_s, for example, a requirement for the function $\hat{h}$ to possess a minimum is

$$\frac{\partial^2 \hat{h}}{\partial \delta y_s^{\,2}} = \frac{\partial^2 H}{\partial y_s^{\,2}} + \delta\lambda_{n+s} \geq 0 \tag{31}$$

and if $\partial^2 H/\partial y_s^{\,2}$ takes on negative values along the reference trajectory, a $\delta\lambda_{n+s} > 0$ at least large enough in magnitude to satisfy (31) will be required to satisfy (29). More generally, the $\delta\lambda_i$, $i = n+1, \ldots, n+\ell$, must be chosen at least large enough in magnitude to insure that the function $\hat{h}$ possesses a minimum.

If the multipliers $\delta\lambda_i$, $i = n+1, \ldots, n+\ell$, are assigned large positive values, corresponding to a restriction to very small step size, the successive approximation process described by our analysis becomes a gradient process. In this case, the process would ultimately approach a weak relative minimum of P', provided that a minimum exists, since its nature is not such as to seek out stationary points of nonminimal character, and the generalized Jacobi necessary condition would automatically be satisfied.

As a practical matter, it seems appropriate to choose values for these multipliers somewhat larger than necessary to satisfy the Weierstrass condition (29), but not so large as to adversely affect the speed of convergence of the process. A conservative, but computationally expensive procedure would be to perform the generalized Jacobi test for the problem of minimizing the approximation (17) at each step of the process, thus insuring that the step-size multipliers have been chosen large enough to exclude generalized conjugate points from the interval $t_o < t < \bar{t}_f$. A more practical procedure, having an element of a gamble, would be merely to check at each step whether or not a decrease in P' has been realized, and to perform the generalized Jacobi test only on the specimen finally obtained after the process has converged. We will discuss the Jacobi test procedure subsequently.

A point neglected in the preceding analysis is the determination of control increments at the terminal point of the reference trajectory $\delta y_{k_f} = \delta y_k(\bar{t}_f)$ which enter the expression (17) whose minimum is sought and which appear consequently in the transversality condition (26). If continuity were required of the control variables y_k, and hence of the δy_k, the control increments at $t = \bar{t}_f$ would be determined by the operation $\min_{\delta y_k} \hat{h}$ just as at interior points of the interval $t_o \leq t < \bar{t}_f$. The introduction of such a continuity requirement at the terminal point is a feasible, if rather arbitrary, means of handling the matter. If, on the other hand, the δy_{k_f} are regarded as free of choice, it follows that they must be chosen so as to provide a minimum of (17). Since the δy_{k_f} appear linearly in (17) with coefficients

$$\sum_{i=1}^{n} \frac{\partial P'}{\partial x_{i_f}} \frac{\partial g_{i_f}}{\partial y_{k_f}} \delta t_f = \frac{\partial H_f}{\partial y_{k_f}} \delta t_f$$

it is possible that no such minimum exists. This situation will definitely arise in the case of a control variable y_s, which is not subject to an inequality constraint of the form (2) if

$$\frac{\partial H_f}{\partial y_{s_f}} \neq 0 \qquad \text{and} \qquad \delta t_f \neq 0.$$

Such considerations suggest the possibility that the control variables chosen for the reference trajectory should not be completely arbitrary but rather should be taken such as to minimize

$$\sum_{i=1}^{n} \frac{\partial P'}{\partial x_i} g_i$$

in the vicinity of the terminal point. The course of action adopted is probably not of key importance computationally, since the effect is local, and

$$H_f = \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{i_f}} g_{i_f}$$

will rapidly approach a minimum in the y_{k_f} in any case.

Computational Procedure for the Penalty Function Process

A possible sequence of calculations is the following:

1) Integrate numerically the system (1) employing the given initial conditions and stored first estimates of the control variables $\overline{y}_k(t)$.

2) Terminate the trajectory at a time $\overline{t}_f$ determined so that P' regarded as a function of t_f along the trajectory attains a minimum, and hence that $\frac{d}{dt_f} P' = 0$. This technique has previously been employed in the gradient/penalty function process.[1,2]

3) Integrate the adjoint system (12) backwards in time employing terminal values $\lambda_i(\bar{t}_f) = \frac{\partial P'}{\partial x_{i_f}}$. Store the initial values $\lambda_i(t_o)$. Calculate the coefficients appearing in the function h (24) during this integration, and select the $\delta\lambda_i$, $i = n+1, \ldots, n+\ell$.

4) Generate by numerical integration a matrix solution of the combined system (9) and (22) with δy_k determined by the operation $\min\limits_{\delta y_k} \hat{h}$, i.e., perform n integrations concurrently with unit matrix initial values of the $\delta\lambda_i$, $i = 1, \ldots, n$, and all δx_i initial values zero. Regenerate the reference trajectory and the adjoint solution concurrently for the purpose of calculating the coefficients of the combined system.

5) By linear algebraic operations, determine the initial values of the $\delta\lambda_i$, $i = 1, \ldots, n$, and the value of δt_f that satisfy the transversality conditions (25) and (26).

6) Employ the $\delta\lambda_i$ initial values so determined for another integration of the combined system (9) and (22). Add the δy_k generated in this solution to the stored $\bar{y}_k$. This furnishes the control functions for a new reference solution.

7) Repeat step 1, starting a new cycle. Compare the value of P' obtained in step 2 of the new cycle with the previous value of P'. The process repeats until decrements in P' become small.

Refinement Process

On account of the penalty function approximation, the process described will converge to a solution whose terminal state variable values differ from those prescribed. For large positive values of the penalty constants K_j, the differences will be small unless the prescribed terminal values are unattainable. The penalty function formulation of the problem serves its purpose in permitting a determination of whether or not this is the case and by providing a scalar measure of convergence - the decrement in P'.

In examining the refinement process described in the following, we assume that the penalty function process has converged closely enough to a minimum that the step size constraints (27) are no longer necessary and that the adjoint variables approximate the multiplier functions of the fixed endpoint problem.

We adopt an expansion for the function P similar to that given by (8) for the function P':

$$P = P(\bar{x}_{m+1_f}, \ldots, \bar{x}_{n_f}, t_f) + \sum_{i=m+1}^{n} \frac{\partial P}{\partial x_{i_f}} \Delta x_{i_f} + \frac{\partial P}{\partial t_f} \delta t_f +$$

$$\frac{1}{2} \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial x_{j_f}} \Delta x_{i_f} \Delta x_{j_f} + \sum_{i=m+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial t_f} \Delta x_{i_f} \delta t_f + \quad (32)$$

$$\frac{1}{2} \frac{\partial^2 P}{\partial t_f^2} \delta t_f^2 + \ldots$$

The fixed state variable terminal conditions will be

$$\bar{x}_{i_f} + \Delta x_{i_f} - \tilde{x}_{i_f} = 0 \qquad i = 1, \ldots, m \qquad (33)$$

Here the $\tilde{x}_{i_f}$ are specified values, the $\bar{x}_{i_f}$ those of the reference trajectory obtained via the penalty function version of the process, and the Δx_{i_f} are the second-order approximations to terminal value increments given by (16). The constraints (33) may be adjoined to (32) by means of additional multipliers μ_i, $i = 1, \ldots, m$, and an approximation sought to the augmented expression which is valid to second order. Approximating the μ_i to zero and first order terms as

$$\mu_i = \bar{\mu}_i + \delta\mu_i \qquad i = 1, \ldots, m \qquad (34)$$

and taking μ_i for the first refinement cycle as the terminal value of the adjoint variable λ_i obtained in the penalty function approximation

$$\bar{\mu}_i = \frac{\partial P'}{\partial x_{i_f}} \qquad i = 1, \ldots, m \qquad (35)$$

we obtain the desired second-order approximation to P as

$$J_o + J_1 + \frac{1}{2} J_2 = P(\bar{x}_{1_f}, \ldots, \bar{x}_{n_f}, \bar{t}_f) + \sum_{i=m+1}^{n} \frac{\partial P}{\partial x_{i_f}} \Big[\xi_{i_f} + (g_{i_f} +$$

$$\sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta x_{j_f} + \sum_{k=1}^{\ell} \frac{\partial g_{i_f}}{\partial y_{k_f}} \delta y_{k_f}\Big) \delta t_f + \tfrac{1}{2}\Big(\sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \bar{g}_{j_f} +$$

$$\sum_{k=1}^{\ell} \frac{\partial g_{i_f}}{\partial y_{k_f}} \dot{\bar{y}}_{k_f} + \frac{\partial g_{i_f}}{\partial t_f}\Big) \delta t_f^{\,2}\Big] + \frac{\partial P}{\partial t_f} \delta t_f + \tfrac{1}{2} \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial x_{j_f}} \times$$

$$\left(\delta x_{i_f} + \bar{g}_{i_f} \delta t_f\right) \left(\delta x_{j_f} + \bar{g}_{j_f} \delta t_f\right) +$$

$$\sum_{i=m+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial t_f} \left(\delta x_{i_f} + \bar{g}_{i_f} \delta t_f\right) \delta t_f + \tfrac{1}{2} \frac{\partial^2 P}{\partial t_f^{\,2}} \delta t_f^{\,2} +$$

$$\sum_{i=1}^{m} \bar{\mu}_i \Big[\xi_{i_f} + \Big(\bar{g}_{i_f} + \sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta x_{j_f} + \sum_{k=1}^{\ell} \frac{\partial g_{i_f}}{\partial y_{k_f}} \delta y_{k_f}\Big) \delta t_f +$$

$$\tfrac{1}{2}\Big(\sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \bar{g}_{j_f} + \sum_{k=1}^{\ell} \frac{\partial g_{i_f}}{\partial y_{k_f}} \bar{y}_{k_f} + \frac{\partial g_{i_f}}{\partial t_f}\Big) \delta t_f^{\,2} + \bar{x}_{i_f} - \tilde{x}_{i_f}\Big] + \tag{36}$$

$$\sum_{i=1}^{m} \delta\mu_i \left(\delta x_{i_f} + \bar{g}_{i_f} \delta t_f + \bar{x}_{i_f} - \tilde{x}_{i_f}\right)$$

In this second-order approximation, δx_{i_f} and δt_f have been regarded as of first order and the difference between ξ_{i_f} and δx_{i_f} as of second order.

The transversality conditions corresponding to δx_{j_f} are

$$\sum_{i=m+1}^{n} \frac{\partial P}{\partial x_{i_f}} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta t_f + \sum_{i=1}^{m} \bar{\mu}_i \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta t_f + \delta\mu_j - \delta\lambda_{j_f} = 0$$

$$j = 1, \ldots, m \tag{37}$$

$$\sum_{i=m+1}^{n} \frac{\partial P}{\partial x_{i_f}} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta t_f + \sum_{i=1}^{m} \bar{\mu}_i \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta t_f +$$

$$\sum_{i=m+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial x_{j_f}} (\delta x_{i_f} + \bar{g}_{i_f} \delta t_f) +$$

$$\frac{\partial^2 P}{\partial x_{j_f} \partial t_f} \delta t_f - \delta \lambda_{j_f} = 0 \qquad j = m+1, \ldots, n \tag{38}$$

to δt_f

$$\sum_{i=m+1}^{n} \frac{\partial P}{\partial x_{i_f}} \left[\bar{g}_{i_f} + \sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta x_{j_f} + \sum_{k=1}^{\ell} \frac{\partial g_{i_f}}{\partial y_{k_f}} \delta y_{k_f} + \left(\sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \bar{g}_{j_f} + \right.\right.$$

$$\left.\left. \sum_{k=1}^{\ell} \frac{\partial g_{i_f}}{\partial y_{k_f}} \dot{y}_{k_f} + \frac{\partial g_{i_f}}{\partial t_f} \right) \delta t_f \right] + \frac{\partial P}{\partial t_f} + \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial x_{j_f}} (\delta x_{i_f} + \bar{g}_{i_f} \delta t_f) \bar{g}_{j_f} +$$

$$\sum_{i=m+1}^{n} \frac{\partial^2 P}{\partial x_{i_f} \partial t_f} (\delta x_{i_f} + 2\bar{g}_{i_f} \delta t_f) + \frac{\partial^2 P}{\partial t_f^2} \delta t_f +$$

$$\sum_{i=1}^{m} \bar{\mu}_i \left[\bar{g}_{i_f} + \sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \delta x_{j_f} + \right. \tag{39}$$

$$\left. \sum_{k=1}^{\ell} \frac{\partial g_{i_f}}{\partial y_{k_f}} \delta y_{k_f} + \left(\sum_{j=1}^{n} \frac{\partial g_{i_f}}{\partial x_{j_f}} \bar{g}_{j_f} + \sum_{k=1}^{\ell} \frac{\partial g_{i_f}}{\partial y_{k_f}} \dot{y}_{k_f} + \frac{\partial g_{i_f}}{\partial t_f} \right) \delta t_f \right] +$$

$$\sum_{i=1}^{m} \delta \mu_i \bar{g}_{i_f} = 0$$

and to $\delta \mu_j$

$$\delta x_{j_f} + \bar{g}_{j_f} \delta t_f + \bar{x}_{j_f} - \tilde{x}_{j_f} = 0 \qquad j = 1, \ldots, m \tag{40}$$

Examination of these relationships establishes that they are linearized versions of the transversality conditions for the original problem. In the case of (39) it is convenient, computationally, to eliminate the $\delta\mu_j$ employing (37), and introduce the $\delta\lambda_{j_f}$ from (38) where appropriate. Since the $\delta\mu_j$ do not appear elsewhere, (37) can then be dropped.

Computational Procedure for the Refinement Process

The procedure for refinement process calculations is generally quite similar to that sketched earlier for the penalty function process, with the following exceptions. Trajectory termination time t_f is taken to be $t_f + \delta t_f$ of the preceding cycle. The adjoint system solution is integrated numerically forward in time, with initial values $\bar{\lambda}_{i_o} + \delta\lambda_{i_o}$, $i = 1, \ldots, n$, of the preceding cycle. Terminal conditions of the equations of variation are (37-40).

Generalized Jacobi Test

The generalized Jacobi test may be applied to the solution obtained after convergence of the process with only slight additional computational effort, since it requires the equations of variation of the Euler-Lagrange equations and the same transversality conditions employed in the successive approximation process. The version of the test dealt with here is applicable only to normal nonsingular extremals. It should be mentioned that a similar restriction applies to the successive approximation process itself. The process fails in the case of a singular subarc appearing in the solution due to the indeterminacy of the Weierstrass condition, and it requires modification in the case of abnormality phenomena.

The generalized Jacobi test matrix is the matrix whose inverse is required in the computation of the $\delta\lambda_{i_o}$, $i = 1, \ldots, n$, and δt_f satisfying, through the equations of variation of the Euler-Lagrange equations (22) and the equations of variation of the basic system (9), the terminal conditions (37-40). Values of the independent variable $t^* > t_o$ at which the matrix becomes singular determine generalized conjugate points. The generalized Jacobi necessary condition is the requirement that there exist no such points in the interior of the interval $t_o < t < t_f$.

For computational test purposes, a succession of times coinciding with values employed in numerical integration are regarded, each in turn, as terminal points, the elements of the test matrix evaluated just as in the successive approximation process, and the determinant computed.

The vanishing of the determinant along the trajectory indicates the existence of a nontrivial solution of the combined equations of variation that satisfies the linearized version of the terminal conditions. According to the usual argument, the nontrivial solution exhibits a corner at the point t^*, at which point the Weierstrass-Erdmann corner conditions are not met, and hence the zero value of the second variation given by the nontrivial solution cannot be the minimum value. The possibility of a negative second variation if $t_o < t^* < t_f$ disqualifies the test extremal as a candidate for a minimizing arc. The generalized Jacobi condition, called the condition of Mayer in the stronger form appropriate to sufficiency proofs, is treated in Refs. 21 and 22.

Treatment of Inequality Constraints on Control Variables

If inequality constraints of the form (2) are operative, an analysis similar to that preceding may be carried out. The difference is that the minimum operation on the function h given by (36) is subject to inequality constraints on the δy_k derived directly from (2) and the control functions $\bar{y}_k(t)$ of the reference trajectory. In this case, the solution of the two-point boundary problem for the minimum of the approximation (17) or (36) cannot be carried out by linear operations, and an important advantage is lost.

In flight performance applications, an inequality constraint on a control variable is usually associated with the appearance of the control variable in the basic system (1) linearly. If the optimal control is bang-bang, it will usually be advantageous to deal with switching times as control parameters, in which case linear methods may then be employed for treatment of the two-point boundary problem arising in the successive approximation process. If not, the occurrence of a singular subarc in the solution is implied and existing numerical schemes fail.

Relationship to Other Computational Techniques

The present scheme has similarities to two existing techniques based upon the Euler-Lagrange equations. The first of these is the generalization of Newton's method studied in Refs. 17-19 but never applied numerically to trajectory problems, to the writers' knowledge. The main difference is in the penalty function approximation and in the use of the step-size constraints employed in the present method to insure the satisfaction of the Legendre-Clebsch necessary condition. It should be noted that the straightforward use of Newton's method may yield a process in which the function h of (24) is maximized rather than minimized over certain intervals, and that the trajectory obtained by the converged process may consequently fail to satisfy the Legendre-Clebsch condition, and hence not furnish a minimum.

The refinement process presently described would be similar to the methods of Refs. 14 and 15 if each new reference trajectory were obtained via the Euler-Lagrange equations, with the solution of the two-point boundary value problem employed only for computation of new multiplier initial values. The limitation of such a technique is that a good first estimate of multiplier initial values is required for convergence. In the present method, the equivalent of this is obtained via the penalty function version of the process which produces the first approximation for the refinement process.

Low Thrust Example

To illustrate this second-order computational technique and compare it with the first-order gradient method, a constant low-thrust transfer between coplanar circular orbits has been calculated. This particular problem, an Earth to Mars transfer, was treated by the present authors in Ref. 2 using first-order steep descent theories. The system of equations governing the motion was given by:

Radial Acceleration

$$\dot{u} = g_1 = \frac{v^2}{R} - A_o\left(\frac{R_o}{R}\right)^2 + \frac{T \sin \theta}{m_o - T/V_e\, t} \tag{41}$$

Circumferential Acceleration

$$\dot{v} = g_2 = -\frac{uv}{R} + \frac{T \cos \theta}{m_o - T/V_e\, t} \tag{42}$$

Radial Velocity

$$\dot{R} = g_3 = u \tag{43}$$

All the initial and final values of the state variables were specified and the transfer time was to be minimized, i.e.,

$$P = t_f \tag{44}$$

The second-order descent process was coded for the IBM 7094 computer. A modified Adams numerical integration scheme was used. The integration step was fixed at two days, dividing most of the trajectories into approximately one hundred intervals.

For the purpose of coding the first-stage penalty function process (but not the second-stage refinement process) the complexity of the numerical calculation was greatly reduced by setting δt_f to zero. Thus, (17) simplified to

$$J_o + J_1 + \tfrac{1}{2} J_2 = P' + \sum_{i=1}^{n} \frac{\partial P'}{\partial x_{i_f}} \xi_{i_f} + \tag{45}$$

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$$\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 P'}{\partial x_{i_f} \partial x_{j_f}} \delta x_{i_f} \delta x_{j_f}$$

The control variable increment $\delta\theta(t)$ was then calculated to obtain the maximum reduction in P' at time t_f. Of course when the trajectory with control variable $\theta + \delta\theta$ was computed, it was not terminated at time t_f but rather at the point of the trajectory with minimum P'. With respect to over-all computational time, this technique represents a compromise between a true second variation calculation and additional programming complexity. For this particular problem, it was advantageous to treat the problem as a fixed time problem when computing $\delta\theta$ using penalty functions.

The initial $\theta(t)$ function corresponded to constant circumferential thrust. This resulted in terminal boundary value errors that averaged 20%. After 6 descent cycles, using the penalty function procedure, the terminal errors averaged 3% with the transfer time at 180 days. After 5 additional cycles of the refinement process, the average boundary value error was reduced to 0.05% and the transfer time had reached its minimum of 193 days. The over-all computer time was two min., thus representing half the computer time required by the first-order gradient program.

Conclusions

The second variation trajectory optimization method described in this paper is appreciably more complicated than the first-order gradient theory. It appears, however, to be economical in the sense of computing time when many optimal trajectories are to be computed. In addition, the generalized Jacobi test may be applied with only slight additional computational effort. As with any computational approach to the solution of optimal trajectories, the effectiveness of the method rests with its judicious application.

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